p-adic Representation Theory (PadicRep)

Shurui Liu, Zhuoni Chi

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Our goal is to formalize basics of p-adic representation theory.
Currently, we focus on $\mathbb{C}$-representations of p-adic groups.

Chapter 1

Td Groups

1.1 Definitions

Let G be a topological group. We say G is a *td group* (also called *l-group*), if G is totally disconnected, Hausdorff (T2), and locally compact.

Definition 1 (Van Dantzig's definition of td group)**.** Let G be a topological group. We say that G has a *compact open subgroup basis* if $\mathcal{N}(1)$ has a basis consisting of compact open subgroups.

Now we want to prove Theorem 13, i.e. the two definitions of td group are equivalent.

Lemma 2. *Let G be a topological group. If G has a compact open subgroup basis, then G is a nonarchimedean group.*

Proof. □

Lemma 3. *Let G be a topological group. Then a neighborhood basis at unit 1 transports to a neighborhood basis at any $x \in G$ by left translation.*

Proof. □

Lemma 4. *A locally compact, Hausdorff, nonarchimedean group is totally disconnected.*

Proof. □

Lemma 5. *Let G be a td group. Then every neighborhood of 1 contains a compact clopen neighborhood of 1.*

Proof. □

Lemma 6. *Let G be a topological group. If K is compact and open subset of G, then there exists $V \in \mathcal{N}(1)$ such that $K \cdot V \subseteq K$.*

Proof. □

Lemma 7. *In Lemma 6, V can be chosen symmetric.*

Proof. □

Lemma 8.

Let G be a td group. Inside a compact clopen neighborhood of 1, there exists a compact open subgroup.

Proof. □

Lemma 9.

Let G be a td group. Every neighborhood of 1 contains a compact open subgroup of G.

Proof. □

Theorem 10 (Van Dantzig (forward direction))**.**

If G is td group (i.e.totally disconnected, locally compact, and Hausdorff topological group), then G has a compact open subgroup basis.

Proof. □

Lemma 11.

If a topological group G has a compact open subgroup basis, then G is locally compact.

Proof. □

Lemma 12.

If a topological group G has a compact open subgroup basis, then G is totally disconnected.

Proof. □

Theorem 13 (Van Dantzig)**.**

Assuming G is Hausdorff, having a compact open subgroup basis is equivalent to being locally compact and totally disconnected.

Proof. □

Theorem 14.

Assuming G is Hausdorff, having a compact open subgroup basis is equivalent to being locally compact and nonarchimedean.

Proof. □

1.2 Example: $\mathrm{GL}_n(K)$ and Its Closed Subgroups

Let K be a non-archimedean local field. We use $M_n(K)$ to denote the ring of square matrices of size n with entries in K, and $\mathrm{GL}_n(K)$ to denote the general linear group.

Lemma 15. *The local field K is T2, locally compact, and totally disconnected.*

Proof. □

Lemma 16 (Topological Properties of Matrices)**.** *For any $n \in \mathbb{N}$, the space $M_n(K)$ is totally disconnected, locally compact, and Hausdorff.*

Proof. This follows from the finite product topology of K, which possesses these properties. □

Lemma 17 (Topological Properties of $\mathrm{GL}_n(K)$)**.** *For any $n \in \mathbb{N}$, the group $\mathrm{GL}_n(K)$ is a totally disconnected, locally compact, Hausdorff topological group.*

Proof. The group $\mathrm{GL}_n(K)$ is an open subset of $M_n(K)$ (as the preimage of $K^\times$ under the continuous determinant map), so it inherits local compactness and total disconnectedness. It is Hausdorff as a subspace of a Hausdorff space. $\square$

Theorem 18. *The group* $\mathrm{GL}_n(K)$ *has a compact open subgroup basis.*

Proof. Since $\mathrm{GL}_n(K)$ is a td group (Lemma 17), the result follows directly from the forward direction of Van Dantzig's theorem (Theorem 10). $\square$

Lemma 19. *Let G be a td group (totally disconnected, locally compact, Hausdorff). Any closed subgroup of G is also a td group.*

Proof. Closed subgroups of locally compact groups are locally compact. Subspaces of totally disconnected Hausdorff spaces are totally disconnected and Hausdorff. $\square$

Chapter 2

Continuous Representations of Groups with Topology

2.1 Continuous representations

Let R be a commutative topological ring, G a group equipped with a topology, and V a topological R-module with continuous addition and scalar multiplication.

Definition 20 (Continuous representations)**.** A *continuous representation* is a representation $\rho : G \to \mathrm{Aut}_R(V)$ whose action map $(g, v) \mapsto \rho(g)v$ is continuous. (In Lean this is defined as a subtype of `Representation` equipped with the continuity proof.)

Definition 21 (Constructor)**.** Given a representation ρ and a proof of continuity of the action map, we obtain a continuous representation.

Lemma 22 (Projections and coercions)**.** *(In dependent type theory/kean) All definitional projections and coercions agree with the data used to build the representation.*

Proof. □

Definition 23 (Action map)**.** We have the action map $G \times V \to V$ associated to a continuous representation.

Lemma 24 (Continuity of the action)**.** *The action map of a continuous representation is continuous by construction.*

Proof. □

Definition 25 (Underlying topological module)**.** The underlying topological R-module is bundled as an object of $\mathrm{TopModuleCat}(R)$.

Definition 26 (From `TopModuleCat`)**.** If M is an object of $\mathrm{TopModuleCat}(R)$ equipped with a continuous action by R-linear maps, this yields a continuous representation.

Lemma 27 (Slice continuity)**.** *For a continuous representation ρ:*

1. *for fixed g, the map $v \mapsto \rho(g)v$ is continuous;*
2. *for fixed v, the map $g \mapsto \rho(g)v$ is continuous;*

3. for fixed g and u, the affine map $v \mapsto \rho(g)v + u$ is continuous.

Proof. □

Lemma 28 (Extensionality)**.** *Two continuous representations are identical exactly when their underlying algebraic representations are identical.*

Proof. □

Lemma 29 (Trivial representations)**.** *The trivial representation is a continuous representation.*

Proof. □

Definition 30 (Restriction)**.** A continuous representation restricts to any subgroup $H \leq G$.

Definition 31 (Descent to a quotient)**.** Assuming G has continuous multiplication, if $N \trianglelefteq G$ and ρ is trivial on N, then ρ descends to a continuous representation of G/N.

2.2 The category of continuous representations

Definition 32 (The category $\mathrm{CRep}_R(G)$)**.** The category $\mathrm{CRep}_R(G)$ is the category of continuous actions of G on topological R-modules, implemented as a category of continuous G-actions in $\mathrm{TopModuleCat}(R)$.

We defined $\mathrm{CRep}_R(G)$ using a more abstract way, i.e. group actions on the category $\mathrm{TopModuleCat}(R)$ for Lean implementation considerations.

Therefore, we need to show that this abstract definition coincide with the natural one, i.e. the category of continuous representations of G.

Definition 33 (Underlying topological module)**.** There is a forgetful functor to $\mathrm{TopModuleCat}(R)$ and an induced map on morphisms.

Definition 34 (Action data)**.** For $V \in \mathrm{CRep}_R(G)$ we extract the underlying action and define the evaluation map $\mathrm{act}(g, v)$.

Lemma 35 (Action continuity)**.** *The action map $G \times V \to V$ is jointly continuous.*

Proof. □

Definition 36 (Forget to algebraic representations)**.** There is a forgetful functor $\mathrm{CRep}_R(G) \to \mathrm{Rep}_R(G)$.

Definition 37 (Associated continuous representation)**.** Every object in $\mathrm{CRep}_R(G)$ yields a continuous representation on its underlying type.

Proof. □

Lemma 38 (Pointwise continuity)**.** *Joint continuity of the action implies continuity of each slice $v \mapsto \rho(g)v$.*

Proof. □

Definition 39 (Action endomorphisms)**.** Each group element acts by a continuous endomorphism of the underlying topological module, and these assemble into a monoid homomorphism $G \to \mathrm{End}(V)$.

Definition 40 (Action object from a representation)**.** A representation with a jointly continuous action determines a continuous action object in TopModuleCat(R).

Definition 41 (Upgrade a representation)**.** A representation with a compatible topology and continuous action determines an object of $\mathrm{CRep}_R(G)$.

Definition 42 (From a continuous representation)**.** A continuous representation (as defined above) can be lifted to an object of $\mathrm{CRep}_R(G)$.

Chapter 3

The Hecke Algebra of a Td Group

3.1 Locally constant functions and distributions on a td space

Let X be a td space (i.e. totally disconnected, locally compact, Hausdorff space).

Definition 43 (Locally constant functions)**.** We write $C^\infty(X)$ for the space of complex-valued locally constant functions on X.

Definition 44 (Compactly supported locally constant functions)**.** We write $C_c^\infty(X)$ for locally constant functions with compact support.

Definition 45 (Distributions)**.** A *distribution* on X is a $\mathbb{C}$-linear functional on $C_c^\infty(X)$.

Lemma 46 (Linearity)**.** *Distributions are additive and $\mathbb{C}$-linear.*

Proof. □

3.1.1 Extension by zero and restriction

Definition 47 (Extension by zero)**.** Given a locally constant function on a subset $Z \subseteq X$, we extend it to X by zero outside Z.

Lemma 48 (Local constancy of the extension)**.** *If Z is clopen, the extension by zero is locally constant.*

Proof. □

Lemma 49 (Compact support of the extension)**.** *If Z is closed and compact, the extension by zero has compact support.*

Proof. □

Definition 50 (Restriction of a distribution)**.** A distribution on X restricts to a subspace Z by evaluating on the extension by zero of test functions on Z.

Definition 51 (Support of a distribution)**.** The support of a distribution F consists of those points x such that every neighborhood of x contains a test function on which F does not vanish.

Lemma 52 (Support is closed)**.** *The support of a distribution is a closed subspace of X.*

Proof. ☐

Lemma 53 (Vanishing on disjoint support)**.** *If the support of a test function is disjoint from the support of F, then F evaluates to zero on that function.*

Proof. ☐

3.2 Distributions on groups

Let G be a totally disconnected, locally compact, Hausdorff topological group.

3.2.1 Translations of functions

Definition 54 (Translations)**.** For $\varphi : G \to \mathbb{C}$ and $g \in G$ we define

$$(R_g\varphi)(x) = \varphi(xg), \qquad (L_g\varphi)(x) = \varphi(g^{-1}x).$$

Lemma 55 (Translation preserves structure)**.** *Right and left translations preserve local constancy and compact support.*

Proof. ☐

Definition 56 (Translation on $C_c^\infty(G)$)**.** Translations of test functions are defined and satisfy the expected identities.

3.2.2 Compactly supported distributions and convolution

Definition 57 (Compactly supported distributions)**.** A compactly supported distribution is a distribution whose support is compact.

Definition 58 (Convolution test function)**.** Given D and φ, the function $x \mapsto D(R_x\varphi)$ is locally constant and compactly supported, and hence defines an element of $C_c^\infty(G)$.

Definition 59 (Convolution)**.** The convolution of F and D is defined by

$$(F * D)(\varphi) = F(x \mapsto D(R_x\varphi)).$$

Lemma 60 (Associativity)**.** *Convolution of compactly supported distributions is associative.*

Proof. ☐

3.2.3 Translation actions on distributions

Definition 61 (Actions on distributions)**.** Left and right translation actions are defined by dualizing the actions on test functions.

3.2.4 Delta distributions and a normalized Haar distribution

Definition 62 (Delta distributions)**.** The delta distribution at g evaluates test functions at g.

Definition 63 (Indicator functions)**.** The indicator function of a compact clopen subset is a compactly supported locally constant function.

Definition 64 (A normalized Haar distribution)**.** We define a compactly supported distribution e_H attached to a compact open subgroup H as the average of a test function over H with respect to the normalized Haar measure on H.

3.2.5 Convolution identities for delta and normalized Haar distributions

Lemma 65 (Idempotence of the normalized Haar distribution)**.** *For a compact open subgroup H, the normalized Haar distribution is idempotent:*

$$e_H * e_H = e_H.$$

Proof. ☐

Lemma 66 (Convolution with delta distributions)**.** *For $g \in G$ and test function φ:*

$$(\delta_g * e_H)(\varphi) = e_H(R_g \varphi), \qquad (e_H * \delta_g)(\varphi) = e_H(L_{g^{-1}} \varphi).$$

If $g \in H$, then $\delta_g * e_H = e_H * \delta_g$.

Proof. ☐

3.2.6 Invariance of the normalized Haar distribution

Lemma 67 (Left/right/bi-invariance of the normalized Haar distribution)**.** *For a compact open subgroup H, e_H is left H-invariant, right H-invariant, and hence bi-H-invariant.*

Proof. ☐

3.2.7 Finite decomposition for bi-invariant distributions

Theorem 68 (Decomposition into convolutions with delta distributions)**.** *If F is compactly supported and bi-H-invariant, then there exist finitely many elements $g_i \in G$ and coefficients $a_i \in \mathbb{C}$ such that*

$$F(\varphi) = \sum_i a_i \left((e_H * \delta_{g_i})(\varphi)\right)$$

for all test functions φ.

Proof. ☐

3.2.8 Local constancy (smoothness)

Definition 69 (Invariant distributions)**.** We say a distribution is left, right, or bi-invariant under an open subgroup if it is fixed by the corresponding translation action.

Theorem 70 (Locally constant distributions)**.** *Let D be a compactly supported distribution on G. The following statements are equivalent:*

1. *D is left-invariant under some compact open subgroup of G;*
2. *D is right-invariant under some compact open subgroup of G;*
3. *D is bi-invariant under some compact open group of G.*

A compactly supported distribution satisfying the above condition is said to be locally constant.

Proof. ☐

3.3 The Hecke algebra

Definition 71 (Hecke algebra)**.** The Hecke algebra $\mathcal{H}(G)$ is the space of compactly supported, locally constant distributions on G, equipped with convolution.

Lemma 72 (Basic structure maps)**.** *(In dependent type theory/lean) The Hecke algebra is a subtype of compactly supported distributions with the expected coercions and extensionality.*

Proof. ☐

Lemma 73 (Additive and scalar structure)**.** *Addition, negation, and scalar multiplication agree with the underlying distribution operations.*

Proof. ☐

Lemma 74 (Convolution product)**.** *Convolution makes $\mathcal{H}(G)$ into a non-unital ring; associativity is inherited from convolution of distributions.*

Proof. ☐

3.4 TODOs

Definition 75. Define idempotented algebra.

Definition 76. Define non-degenerated (also called unital) modules over an idempotented algebra.

Theorem 77. *Hecke algebra is an idempotented algebra.*

Proof. ☐

Definition 78. Define the category of non-degenerated modules over the Hecke algebra.

Theorem 79 (Main theorem)**.** *The category of non-degenerated modules over the Hecke algebra of G is equivalent to the category of smooth $\mathbb{C}$-representations of G.*

Proof. ☐